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of Angle Modulation

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Technical Report No. 710-1
May 5, 1961

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SYSTEMS TECHNIQUES LABORATORY

STANFORD ELECTRONICS LABORATORIES

STANFORD UNIVERSITY · STANFORD, CALIFORNIA

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SUMMARY

A superposition property of angle modulation is derived for a class
of modulating waveforms.

I. A SUPERPOSITION PROPERTY OF ANGLE MODULATION

It is well known that amplitude modulation is a linear operation in the sense that the sidebands resulting from a sum of modulations may be found by linear superposition of the sidebands caused by each individual modulation. Angle modulation,[*] however, is a highly non linear operation, with significant sideband energy arising from the interaction of the various modulation components. Thus, the type of linear superposition of sidebands which holds for amplitude modulation does not hold for angle modulation. This report shows that a different type of linear superposition does hold for an important class of angle modulating waveforms, namely those containing at least one periodic component

Consider the waveform

$$y(t) = \cos [\omega_c t + P(t)] \tag{1}$$

where ω_c is a fixed angular frequency, and $P(t)$ is a periodic angle modulation of period T Since $P(t)$ is periodic, $y(t)$ has a discrete frequency spectrum which may be written

$$y(t) - \sum_n d_n \cos [(\omega_c + \frac{2\pi n}{T})t + \beta_n] \tag{2}$$

A second modulation is now added, giving

$$Y(t) = \cos [\omega_c t + P(t) + f(t)] \tag{3}$$

where $f(t)$ may be random or causal We now find an expression for $Y(t)$ that demonstrates explicitly the effect of $f(t)$ on the discrete spectrum of $y(t)$

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[*] Angle modulation refers to either phase modulation or frequency modulation in its equivalent phase modulating form

Begin by rewriting (3) as

$$Y(t) = \mathrm{Re}\left\{ \epsilon^{j\left[\omega_c t + P(t) + f(\cdot)\right]} \right\}$$

$$= \mathrm{Re}\left\{ \epsilon^{j\left[\omega_c t + f(t)\right]} \epsilon^{jP(t)} \right\} \tag{4}$$

The periodic function $\epsilon^{jP(t)}$ may now be expanded in a complex Fourier series,

$$\epsilon^{jP(t)} = \sum_n C_n \, \epsilon^{j\frac{2\pi n}{T} t} \tag{5}$$

where the C_n are given by

$$C_n = \frac{1}{T} \int_0^T \epsilon^{j\left[P(t) - \frac{2\pi n}{T} t\right]} dt \tag{6}$$

Substitution of the series in (4) and term-by-term multiplication by the first exponential gives

$$Y(t) = \mathrm{Re}\left\{ \sum_n C_n \, \epsilon^{j\left[(\omega_c + \frac{2\pi n}{T}) t + f(t)\right]} \right\} \tag{7}$$

Taking the real part as indicated,

$$Y(t) = \sum_n R_n \cos\left[(\omega_c + \frac{2\pi n}{T}) t + f(t)\right] - I_n \sin\left[(\omega_c + \frac{2\pi n}{T}) t + f(t)\right]$$

$$\tag{8}$$

where

$$R_n \overset{\Delta}{=} \mathrm{Re}\{C_n\}$$

$$I_n \overset{\Delta}{=} \mathrm{Im}\{C_n\} \tag{9}$$

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Use of the trigonometric identity

$$A \cos \theta + B \sin \theta = \sqrt{A^2 + B^2} \cos [\theta + \tan^{-1} (\frac{-B}{A})] \qquad (10)$$

gives

$$Y(t) = \sum_n \sqrt{R_n^2 + I_n^2} \cos [(\omega_c + \frac{2\pi n}{T}) t + \tan^{-1} (\frac{I_n}{R_n}) + f(t)] \qquad (11)$$

The definitions

$$d_n = \sqrt{R_n^2 + I_n^2}$$

$$\beta_n = \tan^{-1} (\frac{I_n}{R_n}) \qquad (12)$$

lead to the desired result,

$$Y(t) = \sum_n d_n \cos [(\omega_c + \frac{2\pi n}{T}) t + \beta_n + f(t)] \qquad (13)$$

Note that when $f(t)$ is identically zero (13) reduces to (2), thereby ensuring that the d_n and β_n are the same in both equations. Equation (13) was previously known to be approximately true,[1,2] but the above proof demonstrates that it is, indeed, exact.

Thus, the periodic modulation $P(t)$ may be considered to divide the carrier into a number of subcarriers separated in frequency by an amount equal to the fundamental frequency of $P(t)$. The added modulation $f(t)$ then angle modulates each of these subcarriers identically. When $f(t)$

1. N. M. Blachman, 'Limiting frequency modulation spectra', Tech. Report No. EDL.-M43, Electronic Defense Lab., Sylvania Electric Products Inc., Mountain View, Calif, April, 1955, pp. 3-5

2. P. R. Karr, 'A note on angle modulation by a mixture of a periodic function and noise', IRE Transactions on Information Theory, Vol. IT-5, September, 1959, pp. 140-143

is causal, the sidebands of the original carrie

linear superposition of the sidebands of the va

viding the contributions at any one frequency

taking their phase relations into account. Wh

power spectrum of $Y(t)$ may be found by simple li

the power spectra of the various subcarriers